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Implicit Finite Differences Method for Solving Couple Nonlinear parabolic system with Constant Coefficients

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Abstract:
In this paper, the implicit finite differences method (IFDM) is used to solve couple of nonlinear parabolic system with constant coefficients (CNPS). At any discrete time t_j the proposed method is transforms the CNPS into a couple nonlinear algebraic system (CNAS), which is solved by applying the predictor- corrector techniques (PCT). This technique reduces the CNAS into a linear system and where the Cholesky decomposition (ChDe) is utilized to solve it at any time t_j. The consistency and the convergence of the method are studied. Two examples are given and are solved using the IFDM to compare their results with the results obtained from solving the same examples but by utilizing the mixed Galerkin - Implicit Differences Methods (MGIDM). The results show that the MGIDM is more accurate than the IFDM.
Keywords: Coupled Nonlinear Parabolic System, Implicit finite Difference Method, Consistency, Convergence.

طريقة الفروقات المنتهية الضمنية لحل زوج نظام مكافئ غير خطي ذات معاملات ثابتة

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مستخلص البحث:
في هذا البحث، ثم استخدام طريقة الفروقات الضمنية لحل زوج نظام مكافئ غير خطي ذات المعاملات الثابتة. عند اي زمن t_j هذه الطريقة تحول زوج نظام مكافئ غير خطي الى نظام جبري غير خطي، والذي تم حله باستخدام تقنية الثنيا والتصحيح، هذه التقنية تحول نظام جبري غير خطي إلى نظام خطي وطريقة جولسكي استخدمت لحل النظام الخطي عند اي زمن t_j. تمت دراسة التأكيد والتقارب للطريقة. ثم اعطاء مثالين وتم حلهما باستخدام طريقة الفروقات المنتهية الضمنية لمقارنة نتائجهما مع النتائج التي نتجت من حل هذين المثالين باستخدام طريقة

خليط كايركن – فروقات ضمنية. اظهرت النتئج ان الطريقة الاخيرة ادق من طريقة الفروقات المنتهية الضمنية.
الكلمات المفتاحية: نظام مكافئ غير خطي مقترن، طريقة الفروق المحدودة الضمنية، الاتساق، التقارب.

1. Introduction

When A parabolic partial differential equations (PPDEs) is a particular type of PDE which is utilized to characterize a large diversity of time-dependent phenomena such a conduction of the heat, particular propagation, and acoustic diffusion (Tadmor, 20212). Due to the particular significance for nonlinear PPDEs (NPPDEs), numerous investigators concerned about the numerical or approximate solution for this kind. In 2019, a hybrid method (Delkhosh & Parand, 2019), the Crank-Nicolson with the Galerkin method (Al-Hawasy & Jawad, 2019)and the homotopy perturbation with transform method suggested (Enodi & Tawfiq, 2019) to solve some types of NPPDEs, while in 2020 it solved by utilizing mixed Haar wavelet collocation method with finite difference (SAleem, Azizi & Hussain, 2020). However, such of these problems solved by other investigators (Esmailzadeh, Alavi & Najafi, 2022),(Datta et al, 2022) and (Somiya etal , 2022) in 2022, all the above methods motivated us to think about solving CNPS by utilizing the IFDM.

This paper starts with the description of the CNPS, the numerical solution (NS) is obtained from the discretization of the continuous CNPS by using the IFDM. At any discrete time t_j the proposed method is transformed the CNPS into a linear form (CLAS) after using the PCT, and then the ChDe method is applied to solved it. Also, the consistency and the convergence of this method is proved. Finally, illustrations examples are presented to solve different problems using the IFDM, the results are given by tables and by figures, and they compared with the MGIDM) to examine the accuracy and efficiency of the proposed method. The results show the MGIDM is more accurate than the IFDM.

2. Description of the CNPS

Let $\Omega = \{\vec{x} = (x_1, x_2) \in R^2 \colon 0 < x_1, x_2 < 1\} \subset R^2$, be the region with boundary $\partial\Omega$, and let $I = [0, T]$, $0 < T < \infty$, then the $Q = \Omega \times I$, then the CNPS are represented as:

$$U_{1t} - \Delta U_1 + U_1 - U_2 = w_1(\vec{x}, t, U_1), \text{ in } Q \quad (1)$$

$$U_{2t} - \Delta U_2 + U_2 + U_1 = w_2(\vec{x}, t, U_2), \text{ in } Q \quad (2)$$

With the initial and the boundary conditions

$$U_i(\vec{x}, 0) = U_i^0(\vec{x}), \text{ in } \Omega, (\forall\ i = 1,2) \quad (3)$$

$$U_i(\vec{x}, t) = 0, \text{ on } \partial\Omega \times I, (\forall\ i = 1,2) \quad (4)$$

Where
$U_1 = U_1(\vec{x},t),\ U_2 = U_2(\vec{x},t) \in C^2(Q),\ \ w_1 = w_1(\vec{x},t,U_1),\ w_2 = w_2(\vec{x},t,U_2)$ are given functions in $C(Q)$ for all $\vec{x} \in \Omega$. The "classical solution" of system ((1)-(4)) is
$\vec{U} = (U_1(\vec{x},t), U_2(\vec{x},t)) \in (C^2(Q))^2, s.t\ \vec{U} = \vec{0}\ on\ \partial\Omega$, for all $\vec{x}$ in Ω.

3. The IFDM of the CNPS

The IFDM (Al-Hawasy & Mansour, 2021) is utilized to find the NS of $\vec{U}^n = (U_1^n, U_2^n)$ of ((1)-(4)), the method begin with discretizing the region of the space variable Ω by the points $(x_{1i}, x_{2j}),\ for\ i,j = 1,2,\dots,N$ with equal spaces $h = {}^1/_N$, and the time with $t_k = t_{k-1} + \Delta t,\ \ k = 1,2,\dots,NT-1$, with $\Delta x_1 = \Delta x_2 = h$. Now, by utilizing the central difference scheme for $x_1,\ x_2$ and forward for t, ((1)-(4)) become

$$\frac{U_1^{k+1}(i,j)-U_1^k(i,j)}{\Delta t} - \frac{U_1^{k+1}(i+1,j)-2U_1^{k+1}(i,j)+U_1^{k+1}(i-1,j)}{h^2} - \frac{U_1^{k+1}(i,j+1)-2U_1^{k+1}(i,j)+U_1^{k+1}(i,j-1)}{h^2} + U_1^{k+1}(i,j) - U_2^{k+1}(i,j) = w_1^k\left(i,j,U_1^{k+1}(i,j)\right) \qquad (5)$$

$$\frac{U_2^{k+1}(i,j)-U_2^k(i,j)}{\Delta t} - \frac{U_2^{k+1}(i+1,j)-2U_2^{k+1}(i,j)+U_2^{k+1}(i-1,j)}{h^2} - \frac{U_2^{k+1}(i,j+1)-2U_2^{k+1}(i,j)+U_2^{k+1}(i,j-1)}{h^2} + U_1^{k+1}(i,j) + U_2^{k+1}(i,j) = w_2^k\left(i,j,U_2^{k+1}(i,j)\right) \qquad (6)$$

putting $r = \frac{\Delta t}{h^2}$, then ((5)-(6)) can writing as

$$(1+\Delta t+4r)U_1^{k+1}(i,j) - r\left[U_1^{k+1}(i+1,j)+U_1^{k+1}(i-1,j)\right] - r\left[U_1^{k+1}(i,j+1)+U_1^{k+1}(i,j-1)\right] - \Delta t U_2^{k+1}(i,j) = U_1^k(i,j) + \Delta t w_1^k\left(i,j,U_1^{k+1}(i,j)\right) \qquad (7)$$

and

$$(1+\Delta t+4r)U_2^{k+1}(i,j) - r\left[U_2^{k+1}(i+1,j)+U_2^{k+1}(i-1,j)\right] - r\left[U_2^{k+1}(i,j+1)+U_2^{k+1}(i,j-1)\right] + \Delta t U_1^{k+1}(i,j) = U_2^k(i,j) + \Delta t w_2^k\left(i,j,U_2^{k+1}(i,j)\right) \qquad (8)$$

Equations ((7)-(8)) represent the IFDM for the CNPS and can be written as the following NAS:

$$A^{k+1}\vec{U} = b\left(\vec{U}^{k+1}\right)$$

where the matrix $A = A^k = \begin{pmatrix} A_1 = A_1^k \\ A_2 = A_2^k \end{pmatrix}$, $A_1 = (a_{ij})_{N^2\times N^2}$ and $A_2 = (\bar{a}_{ij})_{N^2\times N^2}$ with

$$a_{ij}=\begin{cases}1+\Delta t+4r\,, & i=j=1,2,\ldots,N^2\\ -r\,, & j=i+1, i=1,2,\ldots,N^2-1\\ & i\neq (N)l, l=1,2,3\\ -r\,, & i=1,2,3,\ldots,N(N-1)\\ & j=i+N\\ 0 & o.w\end{cases}$$ and

$$\bar{a}_{ij}=\begin{cases}-\Delta t\,, & i=j=1,2,\ldots,N^2\\ 0\,, & o.w\end{cases}$$

4. Consistency of the CNPS

From the Taylor expansion, the following terms are found

$$U_p^{k+1}(i,j)=U_p^k(i,j)+(\Delta t)\frac{\partial U_p^k(i,j)}{\partial t}+\frac{(\Delta t)^2}{2}\frac{\partial^2 U_p^k(i,j)}{\partial t^2}+O(\Delta t^3) \quad (9)$$

$$U_p^{k+1}(i+1,j)-2U_p^{k+1}(i,j)+U_p^{k+1}(i-1,j)=h^2\frac{\partial^2 U_p^k(i,j)}{\partial x_p^2}+O(h^4) \quad (10)$$

$$U_p^{k+1}(i,j+1)-2U_p^{k+1}(i,j)+U_p^{k+1}(i,j-1)=h^2\frac{\partial^2 U_p^k(i,j)}{\partial x_p^2}+O(h^4) \quad (11)$$

Utilizing ((9)-(11)) in ((5)-(6)), to obtain

$$\frac{\partial U_1^k(i,j)}{\partial t}-\frac{\partial^2 U_1^k(i,j)}{\partial x_1^2}-\frac{\partial^2 U_1^k(i,j)}{\partial x_2^2}+U_1^k(i,j)-U_2^k(i,j) - w_1^k\left(i,j,U_1^k(i,j)\right)+(\Delta t)\frac{\partial U_1^k(i,j)}{\partial t}+ \frac{(\Delta t)^2}{2}\frac{\partial^2 U_1^k(i,j)}{\partial t^2}+O(\Delta t)^3=-\frac{\Delta t}{2}\frac{\partial^2 U_1^k(i,j)}{\partial t^2}-(\Delta t)\frac{\partial U_1^k(i,j)}{\partial t}-\frac{\Delta t}{2}\frac{\partial^2 U_1^k(i,j)}{\partial t^2}+ (\Delta t)\frac{\partial U_2^k(i,j)}{\partial t}+\frac{(\Delta t)^2}{2}\frac{\partial^2 U_2^k(i,j)}{\partial t^2}-O(\Delta t)^2+20(h^2) \quad (12)$$

$$\frac{\partial U_2^k(i,j)}{\partial t}-\frac{\partial^2 U_2^k(i,j)}{\partial x_1^2}-\frac{\partial^2 U_2^k(i,j)}{\partial x_2^2}+U_2^k(i,j)+U_1^k(i,j) - w_2^k\left(i,j,U_2^k(i,j)\right)+(\Delta t)\frac{\partial U_2^k(i,j)}{\partial t}+ \frac{(\Delta t)^2}{2}\frac{\partial^2 U_2^k(i,j)}{\partial t^2}+O(\Delta t)^3=-\frac{\Delta t}{2}\frac{\partial^2 U_2^k(i,j)}{\partial t^2}-(\Delta t)\frac{\partial U_1^k(i,j)}{\partial t}-\frac{(\Delta t)^2}{2}\frac{\partial^2 U_1^k(i,j)}{\partial t^2}- (\Delta t)\frac{\partial U_2^k(i,j)}{\partial t}-\frac{(\Delta t)^2}{2}\frac{\partial^2 U_2^k(i,j)}{\partial t^2}-O(\Delta t)+20(h^2)-20(\Delta t)^3$$
(13)

Thus, the IFDM is consistency as h and Δt tend to zero, i.e.

$$U_{1t}^k(i,j)-\Delta U_1^k(i,j)+U_1^k(i,j)-U_2^k(i,j)=w_1^k\left(i,j,U_1^k(i,j)\right) \quad (14)$$

$$U_{2t}^k(i,j)-\Delta U_2^k(i,j)+U_1^k(i,j)+U_2^k(i,j)=w_2^k\left(i,j,U_2^k(i,j)\right) \quad (15)$$

5. Convergence of the CNPS

The convergence of the couple IFDM given by ((7)-(8)) can be obtained by a similar way directly to the convergence for single IFDM through using the Taylor series method (Urena et al, 2021).

6. The Cholesky Decomposition Method

The ChDeM is used to solve the LAS with condition that the coefficients matrix A must be a positive definite, in this method the matrix $A = L.L^T$ can be decomposed into a product of a unique lower triangular matrix L and its transpose (Bacopolous & Chrysssoverghi, 2003). The ChDe can be represented in the following steps:

Step1: $L_{pp} = \left(a_{pp} - \sum_{z=1}^{p-1} L_{pz}^2\right)^{1/2} \quad for\ \ p = 1,2, \dots, N$

Step2: $L_{pq} = \frac{a_{pq} - \sum_{z=1}^{q-1} L_{qz}.L_{pz}}{L_{qq}} \quad for\ q = p+1, \dots, N$

7. Numerical Examples:

In this section numerical examples are carried out to show the efficiency for the presented method in this paper.

Example 7.1: Let I = [0,1], then the CNPS are

$$u_{1t} - \frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_2^2} + u_1 - u_2 = w_1(\vec{x}, t, u_1),\ \ in\ Q$$

$$u_{2t} - \frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial^2 u_2}{\partial x_2^2} + u_1 + u_2 = w_2(\vec{x}, t, u_2)\ \ in\ Q$$

$$u_1(\vec{x}, 0) = u_1^0(\vec{x}) = 0.3\tan(1 - x_2)\sin(2\pi x_1), in\ \Omega$$

$$u_2(\vec{x}, 0) = u_2^0(\vec{x}) = 0\ \ , in\ \Omega$$

$$u_1(\vec{x}, t) = 0,\ \ on\ \partial\Omega \times I$$

$$u_2(\vec{x}, t) = 0,\ \ on\ \partial\Omega \times I$$

Such that the right-hand term $w_1(\vec{x}, t, u_1)$ and $w_2(\vec{x}, t, u_2)$ are given as

$$\begin{aligned} w_1(\vec{x}, t, u_1) = {} & x_1 x_2 t(x_1 - 1)(x_2 - 1) \\ & + 3\sin(2\pi x_1)\left[0.2(tan(x_2 - 1)^2 + 1) \right. \\ & - x_2 tan(x_2 - 1)[0.1 + 0.4\pi^2 \\ & + 0.1\sin(0.3 x_2 tan(x_2 - 1)\sin(2\pi x_1) \\ & \left. + 0.2(tan(x_2 - 1)^2 + 1)]\right] \end{aligned}$$

$$\begin{aligned} w_2(\vec{x}, t, u_2) = {} & x_1 x_2 (x_1 x_2 - x_1 - x_2 \\ & + 1)\left[t\left[\cos\left(t\left(x_1 x_2 (x_1 x_2 - x_1 - x_2 + 1)\right) - 1\right)\right] - 1\right] \\ & + 2t[x_1^2 + x_2^2 - x_1 - x_2] \\ & - [3 x_2 \sin(2\pi x_1) tan(x_2 - 1)]\ /10 \end{aligned}$$

The exact solution (EXS) of the above CNPSCC is

$$u_1(\vec{x}, t) = 0.3 x_2 \tan(1 - x_2)\sin(2\pi x_1)$$

$$u_2(\vec{x}, t) = -x_1 x_2 t(1 - x_1)(1 - x_2)$$

This problem is solved using the IFDM for $M = 9, NT = 20\ \ and\ \ T = 1$, then the NS $\vec{U}^n$ and the EXS $\vec{U}\ at\ x_1\ and\ x_2$ are given at the time

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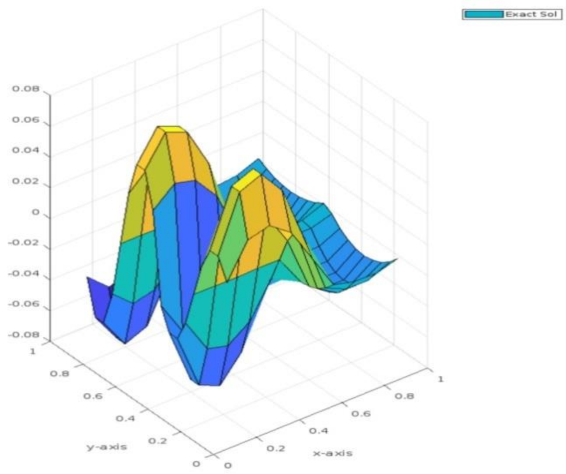
$\hat{t} = 0.5$ in the Table (1) and are shown in Figure (1), the absolute maximum error is (0.0369). It's worth mentioning here that this problem was previously solved using the MIFDM [Ureña F.,2019] for the same values mentioned above, where the maximum error at that time is (0.0038).

Table 7.1: Comparison between the EXS and The APPS Solutions

x_1	x_2	*EXS*	*APPS*	*Absolute error*	x_1	x_2	*EXS*	*APPS*	*Absolute error*
0.1	0.1	0.0222	0.0233	0.0011	0.1	0.1	-0.0040	0.0040	0.0080
0.3	0.1	0.0360	0.0383	0.0023	0.3	0.1	-0.0095	0.0039	0.0134
0.5	0.1	0.0000	0.0020	0.0020	0.5	0.1	-0.0112	-0.0100	0.0012
0.7	0.1	-0.0360	-0.0345	0.0015	0.7	0.1	-0.0095	-0.0208	0.0113
0.9	0.1	-0.0222	-0.0217	0.0005	0.9	0.1	-0.0040	-0.0113	0.0073
0.1	0.3	0.0446	0.0475	0.0029	0.1	0.3	-0.0095	0.0096	0.0191
0.3	0.3	0.0721	0.0784	0.0063	0.3	0.3	-0.0221	0.0100	0.0321
0.5	0.3	0.0000	0.0051	0.0051	0.5	0.3	-0.0262	-0.0228	0.0034
0.7	0.3	-0.0721	-0.0684	0.0037	0.7	0.3	-0.0221	-0.0483	0.0262
0.9	0.3	-0.0446	-0.0432	0.0014	0.9	0.3	-0.0095	-0.0264	0.0169
0.1	0.5	0.0482	0.0517	0.0035	0.1	0.5	-0.0112	0.0107	0.0219
0.3	0.5	0.0779	0.0856	0.0077	0.3	0.5	-0.0262	0.0107	0.0369
0.5	0.5	0.0000	0.0062	0.0062	0.5	0.5	-0.0312	-0.0269	0.0043
0.7	0.5	-0.0779	-0.0735	0.0044	0.7	0.5	-0.0262	-0.0559	0.0297
0.9	0.5	-0.0482	-0.0465	0.0017	0.9	0.5	-0.0112	-0.0305	0.0193
0.1	0.7	0.0382	0.0409	0.0027	0.1	0.7	-0.0095	0.0078	0.0173
0.3	0.7	0.0618	0.0677	0.0059	0.3	0.7	-0.0221	0.0070	0.0291
0.5	0.7	0.0000	0.0049	0.0049	0.5	0.7	-0.0262	-0.0229	0.0033
0.7	0.7	-0.0618	-0.0584	0.0034	0.7	0.7	-0.0221	-0.0455	0.0234
0.9	0.7	-0.0382	-0.0370	0.0012	0.9	0.7	-0.0095	-0.0247	0.0152
0.1	0.9	0.0159	0.0169	0.0010	0.1	0.9	-0.0040	0.0026	0.0066
0.3	0.9	0.0258	0.0279	0.0021	0.3	0.9	-0.0095	0.0017	0.0112
0.5	0.9	0.0000	0.0018	0.0018	0.5	0.9	-0.0112	-0.0101	0.0011
0.7	0.9	-0.0258	-0.0246	0.0012	0.7	0.9	-0.0095	-0.0187	0.0092
0.9	0.9	-0.0159	-0.0155	0.0004	0.9	0.9	-0.0040	-0.0100	0.0060



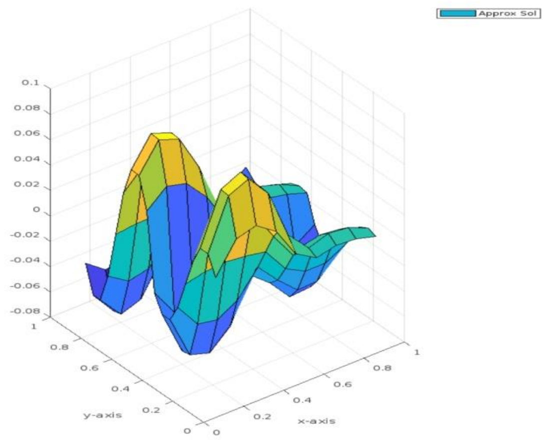


Figure 7.1: shows the EXS and shows the APPS

Example 7.2: Let I= [0,1], then the CNPS are

$$u_{1t}-\frac{\partial^2 u_1}{\partial x_1^2}-\frac{\partial^2 u_1}{\partial x_2^2}+u_1-u_2=w_1(\vec{x},t,u_1),\ \ in\ Q$$

$$u_{2t}-\frac{\partial^2 u_2}{\partial x_1^2}-\frac{\partial^2 u_2}{\partial x_2^2}+u_1+u_2=w_2(\vec{x},t,u_2)\ \ in\ Q$$

$$u_1(\vec{x},0)=u_1^0(\vec{x})=0.2(1-x_1)\sin(2\pi x_2)\,(1-\mathrm{e}^{0.9x_2})\,,in\ \Omega$$

$$u_2(\vec{x},0)=u_2^0(\vec{x})={(1-x_1)(1-x_2)\sin(x_1x_2)}/{5}\quad ,in\ \Omega$$

$$u_1(\vec{x},t)=0,\ \ on\ \partial\Omega\times I$$

$$u_2(\vec{x},t)=0,\ \ on\ \partial\Omega\times I$$

Such that the right-hand term $w_1(\vec{x},t,u_1)$ and $w_2(\vec{x},t,u_2)$ are given as

$$w_1(\vec{x},t,u_1) = -0.2(x_1-1)(x_2-1)\sin(x_1x_2)\sqrt{\mathrm{e}^{-0.11t}} - \sec(t)\sin(2\pi x_2)\big[\mathrm{e}^{0.9x_1}(0.162(x_1-1)+0.36) - 0.2(x_1-1)(\mathrm{e}^{0.9x_1} - 1)[1+4\pi^2+\sec(t)\sin(t) - \sin(0.2(x_1-1)(\mathrm{e}^{0.9x_1} - 1))\sec(t)\sin(2\pi x_2)]\big]$$

$$w_2(\vec{x},t,u_2) = 0.2(x_1-1)(\mathrm{e}^{0.9x_1}-1)\sec(t)\sin(2\pi x_2) - 0.4\sqrt{\mathrm{e}^{-0.11t}}\cos(x_1x_2)[x_2(x_2-1)-x_1(x_1-1)] + 0.2(x_1-1)(x_2 - 1)\sin(x_1x_2)\left[\sqrt{\mathrm{e}^{-0.11t}}\left(1+x_1^2+x_2^2 - 0.2\cos\left((x_1-1)(x_2-1)\sin(x_1x_2)\sqrt{\mathrm{e}^{-0.11t}}\right)\right) - 0.055\mathrm{e}^{-0.11t}/\sqrt{\mathrm{e}^{-0.11t}}\right]$$

The EXS of the above CNPSCC is

$$u_1(\vec{x},t) = 0.2(1-x_1)\sin(2\pi x_2)(1-\mathrm{e}^{0.9x_1})\sec(t)$$

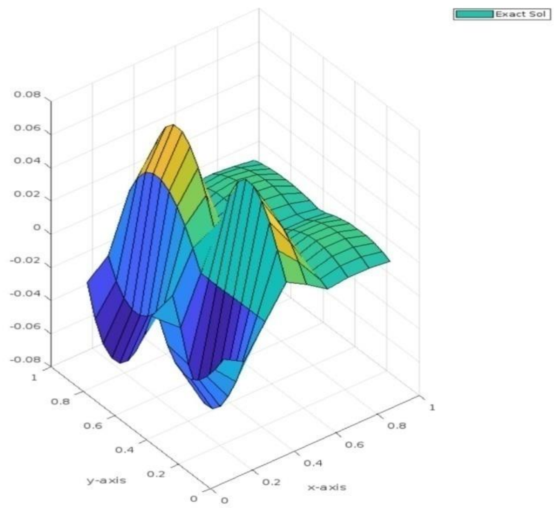
$$u_2(\vec{x},t) = (1-x_1)(1-x_2)\sin(x_1x_2)\sqrt{e^{-0.11t}}\Big/5$$

This problem is solved using the IFDM for $M=9, NT=20$ and $T=1$, then the APPS $\vec{U}^n$ and the EXS $\vec{U}$ at x_1 and x_2 are given at the time $\hat{t}=0.5$ in the Table (2) and are shown in Figure (2), the absolute maximum error is (0.0263). It's worth mentioning here that this problem was previously solved using the MIFDM [Ureña F.,2019] for the same values mentioned above, where the maximum error at that time is (0.0024).

Table 7.2: Comparison between the EXS and The APPS Solutions

x_1	x_2	*EXS*	*APPS*	*Absolute error*	x_1	x_2	*EXS*	*APPS*	*Absolute error*
0.1	0.1	-0.0114	-0.0118	0.0004	0.1	0.1	0.0016	-0.0031	0.0047
0.3	0.1	-0.0291	-0.0301	0.0010	0.3	0.1	0.0037	-0.0087	0.0124
0.5	0.1	-0.0381	-0.0393	0.0012	0.5	0.1	0.0044	-0.0116	0.0160
0.7	0.1	-0.0353	-0.0365	0.0012	0.7	0.1	0.0037	-0.0103	0.0140
0.9	0.1	-0.0167	-0.0173	0.0006	0.9	0.1	0.0016	-0.0043	0.0059
0.1	0.3	-0.0184	-0.0191	0.0007	0.1	0.3	0.0037	-0.0039	0.0076
0.3	0.3	-0.0470	-0.0486	0.0016	0.3	0.3	0.0086	-0.0114	0.0200
0.5	0.3	-0.0616	-0.0634	0.0018	0.5	0.3	0.0102	-0.0156	0.0258

0.7	0.3	-0.0571	-0.0588	0.0017	0.7	0.3	0.0085	-0.0141	0.0226
0.9	0.3	-0.0270	-0.0280	0.0010	0.9	0.3	0.0036	-0.0059	0.0095
0.1	0.5	0.0000	0.0001	0.0001	0.1	0.5	0.0044	0.0044	0.0000
0.3	0.5	0.0000	0.0004	0.0004	0.3	0.5	0.0102	0.0103	0.0001
0.5	0.5	0.0000	0.0007	0.0007	0.5	0.5	0.0120	0.0123	0.0003
0.7	0.5	0.0000	0.0006	0.0006	0.7	0.5	0.0100	0.0102	0.0002
0.9	0.5	0.0000	0.0002	0.0002	0.9	0.5	0.0042	0.0043	0.0001
0.1	0.7	0.0184	0.0194	0.0010	0.1	0.7	0.0037	0.0113	0.0076
0.3	0.7	0.0470	0.0501	0.0031	0.3	0.7	0.0085	0.0288	0.0203
0.5	0.7	0.0616	0.0658	0.0042	0.5	0.7	0.0100	0.0363	0.0263
0.7	0.7	0.0571	0.0609	0.0038	0.7	0.7	0.0082	0.0314	0.0232
0.9	0.7	0.0270	0.0286	0.0016	0.9	0.7	0.0034	0.0131	0.0097
0.1	0.9	0.0114	0.0120	0.0006	0.1	0.9	0.0016	0.0063	0.0047
0.3	0.9	0.0291	0.0309	0.0018	0.3	0.9	0.0036	0.0161	0.0125
0.5	0.9	0.0381	0.0405	0.0024	0.5	0.9	0.0042	0.0205	0.0163
0.7	0.9	0.0353	0.0375	0.0022	0.7	0.9	0.0034	0.0177	0.0143
0.9	0.9	0.0167	0.0177	0.0010	0.9	0.9	0.0014	0.0074	0.0060



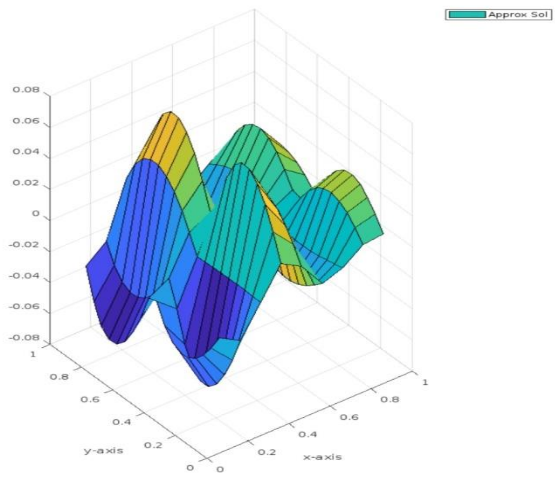


Figure 7.2: shows the EXS and shows the APPS

Conclusion:

The proposed method "IFDM" has been used for solving CNPS. The transformed system of equations (the LAS) was solved by ChDe; this method is very fast for solving LAS. The IFDM was applied easily, and the elements are in numerical or in a full discrete form. Two numerical examples have been used to examine the efficiency and the accuracy of the IFDM. From the results in Tables 1 and 2, the maximum errors between the NS and the EXS in the two examples were (0.0369) and (0.0263), while the maximum errors between them (the NS and the EXS) when these two examples solved using the MGIDM (Al-hawasy & Ibrahim 2024) were (0.0038) and (0.0024). which means the MGIDM is more accurate that the IFDM. It is important to mention here the NS are given at the value of $\hat{t} = 0.5$ to brief the size of the paper, in fact same results with same accuracy were obtained at any value of $\hat{t}$ provide this value belong to I.

Conflicts of Interest: None

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