

Orthogonal Higher Triple Centralizers on Semiprime Γ -Rings

Afrah Mohammad Ibraheem¹

¹Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq

Received: 6/8/2024 Accepted: 30 /8/2024 Published: 1/ 12/ 2024



This work is licensed under a [Creative Commons Attribution-Non-Commercial 4.0 International \(CC BY-NC 4.0\)](https://creativecommons.org/licenses/by-nc/4.0/)

Abstract:

Many specialists in the field of algebra have highlighted many results related to the Γ -ring, in our work we will present the concept of orthogonality on Γ -rings. Let M be a semiprime Γ -ring, $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ are two higher triple centralizers of M , then t and s called orthogonal if for every $x, y \in M, n \in N$, we have $t_n(x)\Gamma M \Gamma s_n(y) = (0) = s_n(y)\Gamma M \Gamma t_n(x)$. Also, in this paper we prove some of lemmas and theorem about orthogonality of the two higher triple centralizers t and s , one of these theorems is: Let

$t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ are two higher triple centralizers of M . If $t_n(x)\Gamma s_n(y) = (0)$ or $s_n(x)\Gamma t_n(y) = (0)$ where t_n and s_n are commuting mapping. Then t_n and s_n are orthogonal.

Keywords: Semiprime gamma ring, Higher centralizer, Higher triple centralizer, orthogonal, highr triple centralizer.

تعامد المركزيات الثلاثية العليا على الحلقات شبه الاولى من النمط Γ

أفراح محمد إبراهيم¹

¹ قسم الرياضيات، كلية التربية، الجامعة المستنصرية، بغداد، العراق

مستخلص البحث:

قد سلط العديد من المتخصصين في مجال الجبر الضوء على العديد من النتائج المتعلقة بالحلقة من النوع- Γ ، وفي عملنا هذا سنقدم مفهوم التعامد على الحلقات من النوع- Γ . لتكن M تكون حلقة شبه أولية من النوع- Γ ، $t = (t_i)_{i \in N}$ ، $s = (s_i)_{i \in N}$ هما مركزيان عليا ثلاثيتان لـ M ، فإن t و s يسمى متعامدتان إذا كان لكل $x, y \in M, n \in N$ لدينا $t_n(x)\Gamma M \Gamma s_n(y) = (0)$ ، وفي هذا البحث أيضاً برهنا بعض القضايا المساعدة والنظريات حول التعامد بين المركزيتين العليا الثلاثيتين t و s ، ومن هذه النظريات: لتكن $t = (t_i)_{i \in N}$ ، $s = (s_i)_{i \in N}$ هما مركزيان عليا ثلاثيتان لـ M ، فإذا كان $t_n(x)\Gamma s_n(y) = (0)$ أو $s_n(x)\Gamma t_n(y) = (0)$ حيث s_n و t_n هما ابداليتين فإن s_n و t_n يكونان متعامدتين.

1. Introduction

The orthogonal derivation on $\square$ -rings was presented by Ozturk and Sapanci (1997). The centralizer on Γ -ring M defined by Hoque and Paul (2011), as an additive mapping T from M to M such that $T(a\alpha b) = T(a)\alpha b$ ($T(a\alpha b) = a\alpha T(b)$) for all $a, b \in M$ and $\alpha \in \Gamma$. Jarullah and Salih (2020) gave concepts orthogonal higher reverse left (right) centralizers of semi-prime rings and they proved some of lemmas and theorems about orthogonally. The concept of higher triple left (resp. right) centralizer of $\square$ -rings introduced by Ibraheem and Salih (2021). In this paper, we define and study orthogonality of a higher triple centralizers for semiprime $\square$ -rings, and prove some theorems about it. We consider a higher triple centralizer of a $\square$ -ring M is a higher triple left and right centralizer, and M is a 2-torsion free semiprime Γ -ring.

2. Results

Lemma 2.1: [1]

Let x, y belong to M , then the statements below are equivalent:

(i) $x\Gamma m\Gamma y = 0$, for all $m \in M$.

(ii) $y\Gamma m\Gamma x = 0$, for all $m \in M$.

(iii) $x\Gamma m\Gamma y + y\Gamma m\Gamma x = 0$, $m \in M$.

If one of these conditions is realized fulfilled, then $x\Gamma y = y\Gamma x = 0$.

Lemma 2.2 : [5]

For $x, y \in M$ if $x\Gamma m\Gamma y + y\Gamma m\Gamma x = 0$, for every $m \in M$. Then $x\Gamma y = y\Gamma x = 0$.

Definition 2.3:

Let $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ be higher triple centralizers of Γ -ring M , then t and s called orthogonal if for every $x, y \in M, n \in N$, we have $t_n(x)\Gamma M\Gamma s_n(y) = (0) = s_n(y)\Gamma M\Gamma t_n(x)$

Where $t_n(x)\Gamma M\Gamma s_n(y) = \sum_{i=1}^n t_i(x)\sigma m\theta s_i(y)$, for every $m \in M$ and $\sigma, \theta \in \Gamma$.

Example 2.4:

Let $M = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in Z \right\}$ be a Γ -ring where $\Gamma = \left\{ \begin{pmatrix} 0 & n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : n \in Z \right\}$. Define $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ two higher triple centralizers of M such that $t_n \left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & n^2 x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$\text{and } s_n \left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & n^2 y & 0 \end{pmatrix}, \quad \text{for all } \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \in$$

M , and $n \in N$. Then t_n and s_n orthogonal.

Lemma 2.5:

Let $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ be two higher triple centralizers of M .

If $t_n(x)IM\Gamma s_n(x) = (0)$, for every $n \in N$, and $x \in M$,

then $t_n(x)IM\Gamma s_n(y) = (0)$, $x, y \in M$.

Proof:

Suppose that, $t_n(x)IM\Gamma s_n(x) = (0)$, that is

$$\sum_{i=1}^n t_i(x)\sigma m\theta s_i(x) = 0, \text{ for all } x, m \in M \text{ and } \sigma, \theta \in \Gamma \quad (1)$$

Commute x by $x+y$ in (1), we get

$$\sum_{i=1}^n t_i(x+y)\sigma m\theta s_i(x+y) = 0,$$

$$\sum_{i=1}^n t_i(x)\sigma m\theta s_i(x) + t_i(x)\sigma m\theta s_i(y) + t_i(y)\sigma m\theta s_i(x) + t_i(y)\sigma m\theta s_i(y) = 0$$

By above assumption and lemma 2.2, we get

$$2 \sum_{i=1}^n t_i(x)\sigma m\theta s_i(y) = 0, \text{ for all } x, y, m \in M \text{ and } \sigma, \theta \in \Gamma.$$

Then, $\sum_{i=1}^n t_i(x)\sigma m\theta s_i(y) = 0$,

So, $t_n(x)IM\Gamma s_n(y) = (0)$.

Lemma 2.6:

Let $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ onto higher triple centralizers of M .

Then $t_n(x)\Gamma s_n(y) + s_n(x)\Gamma t_n(y) = (0)$, for all $x, y \in M$, iff t and s are orthogonal.

Proof:

Let, $t_n(x)\Gamma s_n(y) + s_n(x)\Gamma t_n(y) = (0)$, that is

$$\sum_{i=1}^n t_i(x)\sigma s_i(y) + \sum_{i=1}^n s_i(x)\sigma t_i(y) = 0.$$

Change x by $x\sigma z\theta m$,

$$\sum_{i=1}^n t_i(x\sigma z\theta m)\sigma s_i(y) + \sum_{i=1}^n s_i(x\sigma z\theta m)\sigma t_i(y) = 0,$$

$$\sum_{i=1}^n t_i(x)\sigma t_{i-1}(z)\theta t_{i-1}(m)\sigma s_i(y) +$$

$$\sum_{i=1}^n s_i(x)\sigma s_{i-1}(z)\theta s_{i-1}(m)\sigma t_i(y) = 0, \quad \text{for all } x, y, z, m \in M \text{ and } \sigma, \theta \in \Gamma.$$

By lemma 2.2, we get $t_n(x)IM\Gamma s_n(y) = s_n(x)IM\Gamma t_n(y) = (0)$,

Then, t and s are orthogonal.

Contrariwise,

$t_n(x)IM\Gamma s_n(y) = (0) = s_n(y)IM\Gamma t_n(x)$, that's

$$\sum_{i=1}^n t_i(x)\sigma m\theta s_i(y) = 0, \text{ and } \sum_{i=1}^n s_i(y)\sigma m\theta t_i(x) = 0, \quad \text{for all } x, y \in M \text{ and } \sigma, \theta \in \Gamma.$$

So, by lemma 2.1, we have $\sum_{i=1}^n t_i(x)\sigma s_i(y) = 0$, and

$$\sum_{i=1}^n s_i(x)\sigma t_i(y) = 0$$

Hence, $\sum_{i=1}^n t_i(x)\sigma s_i(y) + \sum_{i=1}^n s_i(x)\sigma t_i(y) = 0$,

Then, $t_n(x)\sigma s_n(y) + s_n(x)\sigma t_n(y) = 0$.

Theorem 2.7:

If $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ are two higher triple centralizers of M , such that t and s are commuting, then

I) t_n and s_n are orthogonal.

$$\text{II) } t_n s_n = 0$$

$$\text{III) } s_n t_n = 0, \text{ and}$$

IV) $t_n s_n + s_n t_n = 0$, are equivalent for every $n \in N$.

Proof:

$I \Leftrightarrow II$

Let t_n and s_n are orthogonal, then

$$t_n(x)\sigma s_n(y) = 0 = s_n(y)\sigma t_n(x),$$

$$\sum_{i=1}^n t_i(x)\sigma s_i(y) = 0 = \sum_{i=1}^n s_i(y)\sigma t_i(x), \forall x, y \in M \text{ and}$$

$\sigma, \theta \in \Gamma$.

$$\text{Take } \sum_{i=1}^n s_i(y)\sigma t_i(x) = 0,$$

$$\sum_{i=1}^n t_i(s_i(y)\sigma t_i(x)) = 0,$$

$$\sum_{i=1}^n t_i(s_i(y))\sigma t_{i-1}(m)\theta t_{i-1}(t_i(x)) = 0,$$

Replace $t_{i-1}(t_i(x))$ by $t_i(s_i(y))$, we have

$$\sum_{i=1}^n t_i(s_i(y))\sigma t_{i-1}(m)\theta t_i(s_i(y)) = 0, \text{ for all } y, m \in M \text{ and } \sigma, \theta \in \Gamma$$

Since M is semiprime, we get

$$\sum_{i=1}^n t_i(s_i(y)) = 0, \text{ for all } y \in M$$

Then, for all $n \in N$ we get, $t_n s_n = 0$.

Conversely,

Let $t_n s_n = 0$, for all $n \in N$

$$t_n(s_n(y)\sigma m\theta x) = 0,$$

$$\sum_{i=1}^n t_i(s_i(y)\sigma s_{i-1}(m)\theta s_{i-1}(x)) = 0, \text{ for all } x, y, m \in M \text{ and } \sigma, \theta \in \Gamma,$$

$$\sum_{i=1}^n t_i(s_i(y))\sigma t_{i-1}(s_{i-1}(m))\theta t_{i-1}(s_{i-1}(x)) = 0$$

Replace $s_i(y)$ by x and $t_{i-1}(s_{i-1}(x))$ by $s_i(y)$, we get

$$\sum_{i=1}^n t_i(x)\sigma t_{i-1}(s_{i-1}(m))\theta s_i(y) = 0 \quad \dots (1)$$

Since t_n and s_n are commuting, we have

$$\sum_{i=1}^n s_i(y)\sigma s_{i-1}(t_{i-1}(m))\theta t_i(x) = 0 \quad \dots (2)$$

From eq.1 and eq. 2, we get the result.

Proof:

$I \Leftrightarrow III$ We get it, similarly $I \Leftrightarrow II$.

Proof:

$I \Leftrightarrow IV$

Let t_n and s_n be orthogonal, then

from II, we have $t_n s_n = 0$,

and by III, we have $s_n t_n = 0$, So, $t_n s_n + s_n t_n = 0$.

Conversely,

suppose that $t_n s_n + s_n t_n = 0$,

$(t_n s_n + s_n t_n)(y \sigma m \theta x) = 0$, for all $x, y, m \in M, \sigma, \theta \in \Gamma$

$t_n(s_n(y \sigma m \theta x)) + s_n(t_n(y \sigma m \theta x)) = 0$,

$\sum_{i=1}^n t_i(s_i(y) \sigma s_{i-1}(m) \theta s_{i-1}(x)) + \sum_{i=1}^n s_i(t_i(y) \sigma t_{i-1}(m) \theta t_{i-1}(x)) = 0$,

$\sum_{i=1}^n t_i(s_i(y)) \sigma t_{i-1}(s_{i-1}(m)) \theta t_{i-1}(s_{i-1}(x)) +$

$\sum_{i=1}^n s_i(t_i(y)) \sigma s_{i-1}(t_{i-1}(m)) \theta s_{i-1}(t_{i-1}(x)) = 0$, for all $x, y, m \in M$, $n \in N, \sigma, \theta \in \Gamma$

Replace $s_i(y)$ by x and $t_i(y)$ by y , we get

$\sum_{i=1}^n t_i(x) \sigma t_{i-1}(s_{i-1}(m)) \theta t_{i-1}(s_{i-1}(x)) +$

$\sum_{i=1}^n s_i(y) \sigma s_{i-1}(t_{i-1}(m)) \theta s_{i-1}(t_{i-1}(x)) = 0$,

Replace $t_{i-1}(s_{i-1}(x))$ by $s_i(y)$ and $s_{i-1}(t_{i-1}(x))$ by $t_i(x)$, we get

$\sum_{i=1}^n t_i(x) \sigma t_{i-1}(s_{i-1}(m)) \theta s_i(y) + \sum_{i=1}^n s_i(y) \sigma s_{i-1}(t_{i-1}(m)) \theta t_i(x) = 0$.

By lemma 2.2, prove the require result.

Theorem 2.8:

Let $t = (t_i)_{i \in N}$ and $s = (s_i)_{i \in N}$ are higher triple centralizers of M . If $t_n(x) \Gamma s_n(y) = (0)$ or $s_n(x) \Gamma t_n(y) = (0)$ where t_n and s_n are commuting mapping, then t_n and s_n are orthogonal.

Proof:

Let $t_n(x) \Gamma s_n(y) = (0)$, then for all $x, y \in M, \sigma \in \Gamma$ and $n \in N$

$\sum_{i=1}^n t_i(x) \sigma s_i(y) = 0$

Replace x by $x \sigma y \theta z$

$\sum_{i=1}^n t_i(x \sigma y \theta z) \sigma s_i(y) = 0$,

$\sum_{i=1}^n t_i(x) \beta t_{i-1}(y) \theta t_{i-1}(z) \sigma s_i(y) = 0$,

Replace $t_{i-1}(y) \theta t_{i-1}(z)$ by $t_{i-1}(m)$ we have

$\sum_{i=1}^n t_i(x) \beta t_{i-1}(m) \sigma s_i(y) = 0$, (1)

Since t_n and s_n are commuting, we get

$\sum_{i=1}^n s_i(y) \beta t_{i-1}(m) \sigma t_i(x) = 0$, for all $y, m, x \in M$ and $\sigma, \beta \in \Gamma$ (2)

Then, by (1) and (2) we have t_n and s_n are orthogonal.

Theorem 2.9:

Let t_n, s_n onto higher centralizers of a 2-torsion free Γ -ring M . If $t_n(x)\Gamma t_n(x) = s_n(x)\Gamma s_n(x)$, then $t_n + s_n$ and $t_n - s_n$ are orthogonal.

Proof:

Since $t_n(x)\Gamma t_n(x) = s_n(x)\Gamma s_n(x)$,

we get for every $x \in M$ and $\sigma \in \Gamma$

$$\begin{aligned} & ((t_n + s_n)\sigma(t_n - s_n) + (t_n - s_n)\sigma(t_n + s_n))(x) = \\ & \sum_{i=1}^n (t_i + s_i)(x)\sigma(t_i - s_i)(x) + (t_i - s_i)(x)\sigma(t_i + s_i)(x) = \\ & \sum_{i=1}^n (t_i(x) + s_i(x))\sigma(t_i(x) - s_i(x)) + (t_i(x) - s_i(x))\sigma(t_i(x) + s_i(x)) = \\ & \sum_{i=1}^n t_i(x)\sigma t_i(x) - t_i(x)\sigma s_i(x) + s_i(x)\sigma t_i(x) - s_i(x)\sigma s_i(x) + t_i(x)\sigma t_i(x) + \\ & t_i(x)\sigma s_i(x) - s_i(x)\sigma t_i(x) - s_i(x)\sigma s_i(x) = \\ & 2(\sum_{i=1}^n t_i(x)\sigma t_i(x) - s_i(x)\sigma s_i(x)) = 0, \text{ since } M \text{ is 2-torsion free } \Gamma\text{-ring, and by} \\ & \text{assumption.} \end{aligned}$$

That's,

$$(t_n + s_n)\sigma(t_n - s_n) + (t_n - s_n)\sigma(t_n + s_n) = 0$$

So, by theorem 2.7 (iv), we prove that $(t_n + s_n)$ and $(t_n - s_n)$ are orthogonal.

3. Acknowledgements

Author of this paper acknowledges support provided by all those who helped accomplish this research and provided advice, guidance, and useful information.

Conflicts of Interest: None**References**

Chakraborty S. & Chandra Paul A. (2008). On Jordan k -Derivations of 2-Torsion Free Prime Γ N-Rings, Punjab University, Journal of Mathematics, Vol. 40, 97-101.

Hoque M. F. & Paul A. C. (2011). On Centralizers of Semiprime Gamma Rings, International Mathematical Forum, Vol. 6, no. 13, 627 - 638.

Ibraheem A. M. & Salih S. M. (2021). Jordan Higher Triple Left Resp.

Right Centralizers of Prime Γ -Rings, Diyala Journal For Pure Science, Vol. 17, no. 3, 25–36, doi: 10.24237/djps.17.03.210-561-icps.

Jarullah F. R. & Salih S. M. (2020). An Orthogonal Higher Reverse Left (Resp. Right) Centralizer On Semiprime Rings, International Journal of Advanced Scientific and Technical Research, Vol. 6, no. 10, doi: 10.26808/rs.st.10V6.03.

Ozturk M. A. & Sapanci M. (1997). Orthogonal symmetric bi-derivation on semi-prime gamma rings, Hacettepe Bulletin of Natural Sciences and Engineering, Vol. 26, 31-4.