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Orthogonal Higher Triple Centralizers on Semiprime Γ-Rings

Afrah Mohammad Ibraheem[1]
[1]*Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq*

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Abstract:
Many specialists in the field of algebra have highlighted many results related to the Γ-ring, in our work we will present the concept of orthogonality on Γ- rings. Let M be a semiprime Γ-ring, $t = (t_i)_{i\in N}$ and $s = (s_i)_{i\in N}$ are two higher triple centralizers of M, then t *and* s called orthogonal if fòr evey $x, y \in M, n \in N$, we have $t_n(x)\Gamma M\Gamma s_n(y) = (0) = s_n(y)\Gamma M\Gamma t_n(x)$. Also, in this paper we prove some of lemmas and theorem about orthogonally of the two higher triple centralizers t and s , one of these theorems is: Let $t = (t_i)_{i\in N}$ and $s = (s_i)_{i\in N}$ are two higher triple centralizers of M. If $t_n(x)\Gamma s_n(y) = (0)$ or $s_n(x)\Gamma t_n(y)$ =(0) where t_n and s_n are commutating mapping. Then t_n and s_n are orthogonal.

Keywords: Semiprime gamma ring, Higher centralizer, Higher triple centrálizer, orthogonal, highr triple centralizer.

تعامد المركزيات الثلاثية العليا على الحلقات شبه الاولية من النمط Γ

أفراح محمد ابراهيم [1]
[1] قسم الرياضيات، كلية التربية، الجامعة المستنصرية، بغداد، العراق

مستخلص البحث:
قد سلط العديد من المتخصصين في مجال الجبر الضوء على العديد من النتائج المتعلقة بالحلقة من النوع- Γ، وفي عملنا هذا سنقدم مفهوم التعامد على الحلقات من النوع- Γ. لتكن M تكون حلقة شبه أولية من النوع- Γ ، $t = (t_i)_{i\in N}$, $s = (s_i)_{i\in N}$ هما مركزيتان عليا ثلاثيتان ل M، فان t و s يسمى متعامدتان إذا كان لكل x,y∈ M, n∈N ، لدينا= $t_n(x)\Gamma M\Gamma s_n(y)$ $(0) = s_n(y)\Gamma M\Gamma t_n(x)$. وفي هذا البحث أيضاً برهنا بعض القضايا المساعدة والنظريات حول التعامد بين المركزيتين العليا الثلاثيتين t و s، ومن هذه النظريات: لتكن $t = (t_i)_{i\in N}$, $s = (s_i)_{i\in N}$ هما مركزيتان عليا ثلاثيتان ل M، فاذا كان $t_n(x)\Gamma s_n(y) = (0)$ او $s_n(x)\Gamma t_n(y)$ =(0) حيث هما ابداليتين فان t_n و s_n يكونان متعامدتين.

1. Introduction

The orthogonal derivation on □-rings was presented by Ozturk and Sapanci (1997). The centralizer on Γ-ring M defind by Hoque and Paul (2011), as an additive mapping T from M to M such that T(aαb) = T(a)αb (T(aαb) = âαT(b)) for all a, b∈M and α∈Γ. Jarullah and Salih (2020) gave concepts orthognal higher reverse left (right) centralizers of semi-prime rings and they proved some of lemmas and theorems about orthogonally. The concept of higher tripe left (resp. right) centralizerof □-rings introduced by Ibraheem and Salih (2021). In this paper, we define and study orthogonality of a higher tripele centralizers for semiprime □-rings, and prove some theorems about it. We consider a higher triple centralize of a □-ring M is a higher triple left and right centralizer, and M is a 2-torsion free semiprime Γ-ring.

2. Results

Lemma 2.1: [1]

Let x, y belong to M, then the statements below are equivalent:

(i) $x\Gamma m\Gamma y = 0$, for all $m \in M$.

(ii) $y\Gamma m\Gamma x = 0$, for all $m \in M$.

(iii) $x\Gamma m\Gamma y + y\Gamma m\Gamma x = 0$, $m \in M$.

If one of these conditions is realize fulfilled, then $x\Gamma y = y\Gamma x = 0$.

Lemma 2.2 : [5]

For x, y ∈M if $x\Gamma m□y + y□m□x = ò$, for every m□M. Then $x□m□y = y□m□x = 0$.

Definition 2.3:

Let $t = (t_i)_{i\in N}$ and $s = (s_i)_{i\in N}$ be higher triple centralizers of Γ-ring M, then t *and* s called orthogonal if for evey $x, y \in M, n \in N$, we have $t_n(x)\Gamma M\Gamma s_n(y) = (0) = s_n(y)\Gamma M\Gamma t_n(x)$

Where $t_n(x)\Gamma M\Gamma s_n(y) = \sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(y)$, for every $m \in M$ and $\sigma, \theta \in \Gamma$.

Example 2.4:

Let $M = \left\{\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in Z\right\}$ be a Γ-ring where $\Gamma = \left\{\begin{pmatrix} 0 & n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : n \in Z\right\}$. Define $t = (t_i)_{i\in N}$ and $s = (s_i)_{i\in N}$ two higher triple centralizers of M such that $t_n\left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & n^2x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $s_n\left(\begin{pmatrix}0 & x & y\\ 0 & 0 & z\\ 0 & 0 & 0\end{pmatrix}\right)=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & n^2y & 0\end{pmatrix}$, $for\ all \begin{pmatrix}0 & x & y\\ 0 & 0 & z\\ 0 & 0 & 0\end{pmatrix}\in M, and\ \ n\in N$. Then $t_n\ and\ s_n$ orthogonal.

Lemma 2.5:

Let $t=(t_i)_{i\in N}$ and $s=(s_i)_{i\in N}$ be two higher triple centralizers of M. If $t_n(x)\Gamma M\Gamma s_n(x)=(0)$, for every $n\in N$, and $x\in M$, then $t_n(x)\Gamma M\Gamma s_n(y)=(0),\ x,y\in M$.

Proof:

Suppose that, $t_n(x)\Gamma M\Gamma s_n(x)=(0)$, that is

$\sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(x)=0$, for all $x,m\in M$ and $\sigma,\theta\in\Gamma$ (1)

Commute x by x+y in (1) , we get

$\sum_{i=1}^{n} t_i(x+y)\sigma m\theta s_i(x+y)=0,$

$\sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(x)+t_i(x)\sigma m\theta s_i(y)+t_i(y)\sigma m\theta s_i(x)+t_i(y)\sigma m\theta s_i(y)=0$

By above assumption and lemma 2.2, we get

$2\sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(y)=0$, for all $x,y,m\in M$ and $\sigma,\theta\in\Gamma$.

Then, $\sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(y)=0,$

So , $t_n(x)\Gamma M\Gamma s_n(y)=(0)$.

Lemma 2.6:

Let $t=(t_i)_{i\in N}$ and $s=(s_i)_{i\in N}$ onto higher triple centralizers of M. Then $t_n(x)\Gamma s_n(y)+s_n(x)\Gamma t_n(y)=(0)$, for all $x,y\in M$, iff $t\ and\ s$ are orthogonal.

Proof:

Let, $t_n(x)\Gamma s_n(y)+s_n(x)\Gamma t_n(y)=0)$, that is

$\sum_{i=1}^{n} t_i(x)\sigma s_i(y)+\sum_{i=1}^{n} s_i(x)\sigma t_i(y)=0.$

Change x by $x\sigma z\theta m$,

$\sum_{i=1}^{n} t_i(x\sigma z\theta m)\ \sigma\ s_i(y)+\sum_{i=1}^{n} s_i(x\sigma z\theta m)\ \sigma\ t_i(y)=0,$

$\sum_{i=1}^{n} t_i(x)\sigma t_{i-1}(z)\theta t_{i-1}(m)\ \sigma s_i(y)+\sum_{i=1}^{n} s_i(x)\sigma s_{i-1}(z)\theta s_{i-1}(m)\ \sigma t_i(y)=0,$ for all $x,y,z,m\in M$ and $\sigma,\theta\in\Gamma$.

By lemma 2.2, we get $t_n(x)\Gamma M\Gamma s_n(y)=s_n(x)\Gamma M\Gamma t_n(y)=(0)$,

Then, t and s are orthogonal.

Contrariwise,

$t_n(x)\Gamma M\Gamma s_n(y)=(0)=s_n(y)\Gamma M\Gamma t_n(x)$, that's

$\sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(y)=0,\ and\ \sum_{i=1}^{n} s_i(y)\sigma m\theta\, t_i(x)=0,$ for all $x,y\in M$ and $\sigma,\theta\in\Gamma$.

So, by lemma 2.1, we have $\sum_{i=1}^{n} t_i(x)\sigma s_i(y) = 0$, and
$\sum_{i=1}^{n} s_i(x)\sigma\, t_i(y) = 0$
Hence, $\sum_{i=1}^{n} t_i(x)\sigma s_i(y) + \sum_{i=1}^{n} s_i(x)\sigma t_i(y) = 0$,
Then, $t_n(x)\Gamma s_n(y) + s_n(x)\Gamma t_n(y) = (0)$.

Theorem 2.7:

If $t = (t_i)_{i\in N}$ and $s = (s_i)_{i\in N}$ are two higher triple centralizers of M, such that t and s are commutng, then

I) t_n and s_n are orthogonal.

II) $t_n s_n = 0$

III) $s_n t_n = 0$, and

IV) $t_n s_n + s_n t_n = 0$, are equivalent for every $n \in N$.

Proof:

I ⇔ II

Let t_n and s_n are orthogonal, then
$t_n(x)\Gamma M\Gamma s_n(y) = (0) = s_n(y)\Gamma M\Gamma t_n(x)$,
$\sum_{i=1}^{n} t_i(x)\sigma m\theta s_i(y) = 0 = \sum_{i=1}^{n} s_i(y)\sigma m\theta\, t_i(x)$, $\forall\, x, y \in M$ and $\sigma, \theta \in \Gamma$.
Take $\sum_{i=1}^{n} s_i(y)\sigma m\theta\, t_i(x) = 0$,
$\sum_{i=1}^{n} t_i(s_i(y)\sigma m\theta\, t_i(x)) = 0$,
$\sum_{i=1}^{n} t_i(s_i(y))\sigma t_{i-1}(m)\theta\, t_{i-1}(t_i(x)) = 0$,
Replace $t_{i-1}(t_i(x))$ by $t_i(s_i(y))$, we have
$\sum_{i=1}^{n} t_i(s_i(y))\sigma t_{i-1}(m)\theta\, t_i(s_i(y)) = 0$, for all $y, m \in M$ and $\sigma, \theta \in \Gamma$
Since M is semiprime, we get
$\sum_{i=1}^{n} t_i(s_i(y)) = 0, \textit{for all } y \in$ M
Then, for all n ∈ N we get, $t_n s_n = 0$.

Conversely,

Let $t_n s_n = 0$, for all n ∈ N
$t_n(s_n(y\sigma m\theta x)) = 0$,
$\sum_{i=1}^{n} t_i(s_i(y)\sigma s_{i-1}(m)\theta s_{i-1}(x)) = 0, \textit{for all}$ x, $y, m \in M$ and $\sigma, \theta \in \Gamma$,
$\sum_{i=1}^{n} t_i(s_i(y))\sigma\, t_{i-1}(s_{i-1}(m))\theta\, t_{i-1}(s_{i-1}(x)) = 0$
Replace $s_i(y)$ by x and $t_{i-1}(s_{i-1}(x))$ by $s_i(y)$, we get
$\sum_{i=1}^{n} t_i(x)\sigma\, t_{i-1}(s_{i-1}(m))\,\theta s_i(y) = 0 \quad \dots (1)$
Since t_n *and* s_n are commuting, we have
$\sum_{i=1}^{n} s_i(y)\sigma\, s_{i-1}(t_{i-1}(m))\theta t_i(x) = 0 \quad \dots (2)$
From eq.1 and eq. 2, we get the result.

Proof:

I ⇔ III We get it, similarly I ⇔ II.

Proof:

I ⇔IV

Let t_n and s_n be orthogonal, then
from II, we have $t_n s_n = 0$,
and by III, we have $s_n t_n = 0$, So, $t_n s_n + s_n t_n = 0$.

Conversely,
suppose that $t_n s_n + s_n t_n = 0$,
$(t_n s_n + s_n t_n)(y\sigma m\theta x) = 0$, for all x , y, m ∈ M, $\sigma, \theta \in \Gamma$
$t_n(s_n(y\sigma m\theta x)) + s_n(t_n(y\sigma m\theta x)) = 0$,
$\sum_{i=1}^{n} t_i(s_i\,(y)\sigma s_{i-1}(m)\theta s_{i-1}(x)) + \sum_{i=1}^{n} s_i(t_i\,(y)\sigma t_{i-1}(m)\theta t_{i-1}(x)) = 0$,
$\sum_{i=1}^{n} t_i(s_i\,(y))\sigma t_{i-1}(\,s_{i-1}(m))\theta t_{i-1}(\,s_{i-1}(x)) + \sum_{i=1}^{n} s_i(t_i\,(y))\sigma\, s_{i-1}(t_{i-1}(m))\theta s_{i-1}(\,t_{i-1}(x)) = 0$, for all x, y, m ∈M, n∈N, $\sigma, \theta \in \Gamma$
Replace $s_i(y)$ by x and $t_i(y)$ by y , we get
$\sum_{i=1}^{n} t_i(x)\sigma t_{i-1}(s_{i-1}(m))\theta t_{i-1}(s_{i-1}(x)) + \sum_{i=1}^{n} s_i(y)\sigma s_{i-1}(t_{i-1}(m))\theta s_{i-1}(t_{i-1}(x)) = 0$,
Replace $t_{i-1}(\,s_i(x)$ by $s_i(y)$ *and* $s_{i-1}(t_{i-1}(x)$ *by* $t_i(x)$, we get
$\sum_{i=1}^{n} t_i(x)\sigma t_{i-1}(s_{i-1}(m)\theta\, s_i(y) + \sum_{i=1}^{n} s_i(y)\sigma s_{i-1}(t_{i-1}(m))\theta\, t_i(x) = 0$.
By lemma 2.2, prove the require result.

Theorem 2.8:

Let $t = (t_i)_{i\in N}$ and $s = (s_i)_{i\in N}$ are higher triple centralizers of M. If $t_n(x)\Gamma s_n(y) = (0)$ or $s_n(x)\Gamma t_n(y) = (0)$ where t_n and s_n are commutating mapping, then t_n and s_n are orthogonal.

Proof:

Let $t_n(x)\Gamma s_n(y) = (0)$, then for all x, y∈M, $\sigma \in \Gamma$ and n∈N
$\sum_{i=1}^{n} t_i(x)\sigma\, s_i(y) = 0$
Replace x by $x\sigma y\theta z$
$\sum_{i=1}^{n} t_i(x\beta y\theta z)\sigma s_i(y) = 0$,
$\sum_{i=1}^{n} t_i(x)\beta\, t_{i-1}(y)\theta t_{i-1}(z)\sigma\, s_i(y) = 0$,
Replace $t_{i-1}(y)\theta t_{i-1}(z)$ *by* $t_{i-1}(m)$ we have
$\sum_{i=1}^{n} t_i(x)\beta\, t_{i-1}(m)\sigma\, s_i(y) = 0$, (1)
Since t_n and s_n are commutating, *we get*
$\sum_{i=1}^{n} s_i(y)\beta\, t_{i-1}(m)\sigma\, t_i(x) = 0$, for all y, m,x,y ∈M and $\sigma, \beta \in \Gamma$ (2)
Then, by (1) and (2) we have t_n and s_n are orthogonal.

Theorem 2.9:

Let t_n , s_n onto higher centralizers of a 2-torsion free Γ-ring M. $If\ t_n(x)\Gamma t_n(x) = s_n(x)\Gamma s_n(x)$, then $t_n + s_n$ and $t_n - s_n$ are orthogonal.

Proof:

Since $t_n(x)\Gamma t_n(x) = s_n(x)\Gamma s_n(x)$,
we get for every $x \in M$ and $\sigma \in \Gamma$
$((t_n + s_n)\sigma(t_n - s_n) + (t_n - s_n)\sigma(t_n + s_n))(x) =$
$\sum_{i=1}^{n}(t_i + s_i)(x)\sigma(t_i - s_i)(x) + (t_i - s_i)(x)\sigma(t_i + s_i)(x) =$
$\sum_{i=1}^{n}\big(t_i(x) + s_i(x)\big)\sigma\big(t_i(x) - s_i(x)\big) + \big(t_i(x) - s_i(x)\big)\sigma\big(t_i(x) + s_i(x)\big) =$
$\sum_{i=1}^{n} t_i(x)\sigma t_i(x) - t_i(x)\sigma s_i(x) + s_i(x)\sigma t_i(x) - s_i(x)\sigma s_i(x) + t_i(x)\sigma t_i(x) +$
$t_i(x)\sigma s_i(x) - s_i(x)\sigma t_i(x) - s_i(x)\sigma s_i(x) =$
$2(\sum_{i=1}^{n} t_i(x)\sigma t_i(x) - s_i(x)\sigma s_i(x))= 0$, since M is 2-torsion free Γ-ring , and by assumption.
That's,
$(t_n + s_n)\sigma(t_n - s_n) + (t_n - s_n)\sigma(t_n + s_n) = 0$
So, by theorem 2.7 (iv), we prove that $(t_n + s_n)$ and $(t_n - s_n)$ are orthogonal.

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Conflicts of Interest: None

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